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Spectral characterization of the optional quadratic variation process

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Abstract

In this paper we show how the periodogram of a semimartingale can be used to characterize the optional quadratic variation process.

Keywords: Semimartingale; Quadratic variation; Periodogram

1. Introduction and notation

As is well known in the statistical analysis of time series in discrete or continuous time, the periodogram can be used for estimation problems in the frequency domain. It follows from the results of the present paper that the periodogram can also be used to estimate the variance of the innovations of a time series in continuous time. Usually, in statistical problems this variance is assumed to be known, since it can be estimated with probability one, given the observations on any nonempty interval in a number of cases (see, for instance, Dzhaparidze and Yaglom (1983, Theorem 2.1).

A fundamental result in another approach is now known as Levy's theorem, which states that the variance of a Brownian motion can be obtained as the limit of the sum of squares of the increments by taking finer and finer partitions. This result has been generalized by Baxter (1956), who showed a similar result for more arbitrary Gaussian processes (that need not to be semimartingales) and to the case where the process under consideration is a semimartingale by Doleans-Dade (1969), who obtained a characterization of the quadratic variation. See also theorem VIII.20 of Dellacherie and Meyer (1980) or Theorem 4 on p. 55 of Liptser and Shiryaev (1989). Related work on the so-called convergence of order p has been conducted by Lepingle (1976).

In the present paper we take a different viewpoint towards the quadratic variation process (more in the spirit of the Theorem 2.1 of Dzhaparidze and Yaglom (1983) and it is our purpose to show that the periodogram of a semimartingale can be used as

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a statistic to estimate its quadratic variation process. We thus obtain an alternative characterization of this process as compared to, for instance, Doleans-Dade's.

The rest of this section is devoted to the introduction of some notation.

Let $(\Omega, \mathscr{F}, F, P)$ be complete filtered probability space and X real valued semimartingale defined on it. X_0 is assumed to be zero. Let $(\Lambda, \mathscr{L}, Q)$ be an additional probability space. Consider the product $\Omega \times \Lambda$ and endow it with the product σ-algebra $\mathscr{F} \otimes \mathscr{L}$ and the product measure $P \otimes Q$. Identify $\mathscr{F}$ with $\mathscr{F} \otimes \{\emptyset, \Lambda\}$ as a σ-algebra on $\Omega \otimes \Lambda$.

Define for each finite stopping time T and each real number λ the *periodogram* of X evaluated at T by

$$I_T(X;\lambda) = \left| \int_{[0,T]} e^{i\lambda t} \, dX_t \right|^2 .$$

An application of Ito's formula gives

$$I_T(X;\lambda) = 2\mathrm{Re} \int_{[0,T]} \int_{[0,T)} e^{i\lambda(t-s)} \, dX_s \, dX_t + [X]_T. \tag{1.1}$$

On the other hand one can use partial integration to rewrite the periodogram as

$$\begin{aligned} I_T(X;\lambda) &= |e^{i\lambda T} X_T - i\lambda \int_{[0,T]} e^{i\lambda t} X_t \, dt|^2 \\ &= X_T^2 + X_T \int_{[0,T]} i\lambda (e^{i\lambda(T-t)} - e^{-i\lambda(T-t)}) X_t \, dt + \lambda^2 | \int_{[0,T]} e^{i\lambda t} X_t \, dt|^2 . \end{aligned} \tag{1.2}$$

Let $\xi : \Lambda \to \mathbb{R}$ be a real random variable with an absolutely continuous distribution (w.r.t. Lebesgue measure), that has a density G, which is assumed to be symmetric around zero and consider for any positive real number L the quantity

$$E[I_T(X;L\xi)|\mathscr{F}] = E_\xi I_T(X;L\xi) := \int_{\mathbb{R}} I_T(X;Lx) G(x) \, dx . \tag{1.3}$$

In Protter (1990, pp. 159 and 160), conditions can be found under which the interchanging of the integration order in (1.3) is allowed to obtain

$$E_\xi I_T(X;L\xi) = 2 \int_{[0,T]} \int_{[0,t)} g(L(t-s)) \, dX_s \, dX_t + [X]_T, \tag{1.4}$$

where g is the (real) characteristic function of ξ.

These conditions that are valid under quite general circumstances seem to be overly restrictive for the present situation. A much simpler condition is $E\xi^2 < \infty$. Then it follows from Eq. (1.2) that Fubini's theorem can be applied, since the integrals involved there have a pathwise meaning and X is a.s. bounded over $[0, T]$. However, there are also circumstances from which it is clear that even this condition is superfluous. For example, if X is a discrete time process, or more general a process of bounded variation, then it follows from Eq. (1.1) that interchanging the integration order is always allowed. Throughout the paper we assume that Eq. (1.4) holds. Our purpose is to study the behaviour of $E_\xi I_T(X; L\xi)$ for $L \to \infty$. To that end we investigate this quantity for a number of distinguished cases in the next sections.

Remark. As is well known, g is a continuous function and for all $s < t$, it holds that $g(L(t-s)) \to 0$ for $L \to \infty$, in view of the Riemann–Lebesgue lemma (cf. Feller, 1971), and (of course) $|g(x)| \leq 1$ for all real x.

2. Semimartingales of bounded variation

Throughout this section we assume that X is a process of bounded variation over each finite interval. Denote by $\|X\|_t$ the variation of the process X over the interval $[0, t]$ (t may be replaced by a finite stopping time T). In this case we obtain from (1.4),

$$|E_\xi I_T(X; L\xi) - [X]_T| \leq 2 \int_{[0,T]} \int_{[0,t)} |g(L(t-s))| \,\mathrm{d}\|X\|_s \,\mathrm{d}\|X\|_t \tag{2.1}$$

Since $\lim_{L\to\infty} g(Lx) = \delta(x)$ (with $\delta(x) = 1$, if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$), an application of Lebesgue's dominated convergence theorem yields that the right-hand side of (2.1) converges (almost surely) to

$$2 \int_{[0,T]} \int_{[0,t)} \delta(t-s) \,\mathrm{d}\|X\|_s \,\mathrm{d}\|X\|_t.$$

But this is equal to zero, since $\delta(t-s) = 0$ for all $s < t$, whence the following result.

Proposition 2.1. *Let X be a semimartingale of bounded variation, T a finite stopping time and ξ a real random variable, independent of $\mathscr{F}$, which has a density on the real line. Then almost surely for $L \to \infty$*

$$E_\xi I_T(X; L\xi) \to [X]_T.$$

Remark. Notice the similarity of the above statement with formula (1.5) on p. 620 of Feller (1971), if we take the case where ξ has a uniform distribution on $[-1, +1]$, and where X is a piecewise constant process.

3. Semimartingales with bounded jumps

In the first part of this section we assume that X is an arbitrary semimartingale. Starting point for our analysis is again Eq. (1.4). Consider now the process Y^L defined by

$$Y^L = \int_{[0,.]} \int_{[0,t)} g(L(t-s)) \,\mathrm{d}X_s \,\mathrm{d}X_t. \tag{3.1}$$

First we will give an upper bound for the absolute value of the inner integral in (3.1). Thereto we have to introduce some notation. First we need the moduli of continuity of X over an interval I:

$$W_X(I) = \sup\{|X_u - X_v|: u, v \in I\}.$$

Furthermore we have $X_t^* = \sup\{|X_s|: s \le t\}$. Next for any function (or process) Z, we denote by $V(Z; I)$ the total variation of Z over the interval I. (Notice that $V(X; I) = \infty$ for continuous local martingales X and for any interval I, except when $\int_I \mathrm{d}\langle X\rangle$ is zero.)

Lemma 3.1. *Let X be a semimartingale and g a real characteristic function of an absolutely continuous distribution. Then for all $\varepsilon > 0$ and $t > 0$, the following estimate is valid almost surely:*

$$\left|\int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s\right| \le W_X[t-\varepsilon, t)(1 + V(g;[0, L\varepsilon])) + X_t^*[|g(L\varepsilon)| + V(g;[L\varepsilon, Lt])] \tag{3.2}$$

as well as the coarser estimate

$$\left|\int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s\right| \le 2W_X[t-\varepsilon, t)\,V(g;[0,\infty)) + 2X_t^*\,V(g;[L\varepsilon,\infty)). \tag{3.3}$$

Proof. To avoid trivialities, we can assume that both $V(g;[0, L\varepsilon])$ and $V(g;[L\varepsilon, Lt])$ are finite. Consider first $\int_{(t-\varepsilon,t)} g(L(t-s))\,\mathrm{d}X_s$. (If $\varepsilon > t$, then we interpret the integral by extending the definition of X to the negative real line and setting $X_t = 0$ for $t < 0$.) Integration by parts together with the fact that g is continuous yields that this integral is equal to

$$X_{t-} - g(L\varepsilon)X_{t-\varepsilon} - \int_{(t-\varepsilon,t)} (X_s - X_{t-\varepsilon})\,\mathrm{d}g(L(t-s)) - X_{t-\varepsilon}\int_{(t-\varepsilon,t)} \mathrm{d}g(L(t-s))$$

$$= X_{t-} - X_{t-\varepsilon} - \int_{(t-\varepsilon,t)} (X_s - X_{t-\varepsilon})\,\mathrm{d}g(L(t-s)).$$

Hence,

$$\left|\int_{(t-\varepsilon,t)} g(L(t-s))\,\mathrm{d}X_s\right| \le |X_{t-} - X_{t-\varepsilon}| + \sup_{t-\varepsilon<s<t} |X_s - X_{t-\varepsilon}| \int_{(t-\varepsilon,t)} \mathrm{d}\|g\|(L(t-s))$$

$$\le W_X[t-\varepsilon, t)(1 + V(g;[0, L\varepsilon])).$$

Consider now the integral over $[0, t-\varepsilon]$. Using again integration by parts, we obtain

$$\left|\int_{[0,t-\varepsilon]} g(L(t-s))\,\mathrm{d}X_s\right| \le |g(L\varepsilon)X_{t-\varepsilon}| + X_{t-\varepsilon}^* \int_{[0,t-\varepsilon]} \mathrm{d}\|g\|(L(t-s))$$

$$\le X_{t-\varepsilon}^*(|g(L\varepsilon)| + V(g;[\varepsilon, Lt])).$$

Putting the above two estimates together, we obtain the first statement of the lemma. The second one is a simple consequence, since $V(g;[0,\infty)) \geq 1$, $V(g;[0,\infty)) \geq V(g;[0,L\varepsilon])$ and $V(g;[L\varepsilon,\infty)) \geq |g(L\varepsilon)|$. □

Next we prove the analog of Proposition 2.1 for the case of a semimartingale X, that has bounded jumps. The main result of this section is the following.

Proposition 3.2. *Let X be a semimartingale that has bounded jumps and let the function g be of bounded variation over $[0,\infty)$. Let T be a finite stopping time. Then*

$$E_\xi I_T(X;L\xi) \to [X]_T$$

in probability, for $L \to \infty$.

Proof. There exists a decomposition of X as $X = Z + M$, where M is a local martingale such that M also has bounded jumps, $\sup|\Delta M_t| \leq 1$ say, and Z is a process of bounded variation. This follows from the decomposition theorem for local martingales. In particular, M is locally square integrable (cf. Dellacherie and Meyer, 1980, VI.85). Use this decomposition to write Y_T^L from Eq. (3.1) as the sum of two terms. These two are

$$Y_T^L(X,Z) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s\,\mathrm{d}Z_t, \tag{3.4}$$

$$Y_T^L(X,M) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s\,\mathrm{d}M_t, \tag{3.5}$$

Since the process Z is of bounded variation, it follows that $|Y_T^L(X,Z)|$ is bounded above by

$$\int_{[0,T]} \left| \int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s \right| \mathrm{d}\|Z\|_t.$$

First we observe that for each t, $\lim_{\varepsilon\downarrow 0} W_X[t-\varepsilon,t) = 0$ a.s. Indeed, choose a set Ω_0 with probability one, such that for all sample paths defined on this set it holds that $W_X[0,t) < \infty$ and that they all have left limits at t. Take for $\omega \in \Omega_0$ a sample path $X = X(\omega)$. Let $\delta > 0$. Then there is $\varepsilon(\omega) > 0$ such that $|X_u(\omega) - X_{t-}(\omega)| < \delta$ for all u such that $t - \varepsilon(\omega) \leq u < t$ by the fact that left limits exist at t. The same holds for v taken from the same interval. Hence, the triangle inequality gives for such u and v that $|X_u(\omega) - X_v(\omega)| \leq |X_u(\omega) - X_{t-}(\omega)| + |X_{t-}(\omega) - X_v(\omega)| < 2\delta$. Stated otherwise, $W_{X(\omega)}[t-\varepsilon(\omega),t) \leq 2\delta$. Write now $W_{X(\omega)}[t-\varepsilon,t) = 1_{\{\varepsilon\leq\varepsilon(\omega)\}} W_{X(\omega)}[t-\varepsilon,t) + 1_{\{\varepsilon>\varepsilon(\omega)\}} W_{X(\omega)}[t-\varepsilon,t) \leq 2\delta + 1_{\{\varepsilon>\varepsilon(\omega)\}} W_{X(\omega)}[0,t)$. For $\varepsilon\downarrow 0$ the indicator on the right-hand side of the last inequality becomes zero, so we obtain that $\limsup_{\varepsilon\downarrow 0} W_{X(\omega)}[t-\varepsilon,t) \leq 2\delta$. Since $\delta > 0$ is arbitrary, we have for all $\omega \in \Omega_0$ that $\lim_{\varepsilon\downarrow 0} W_{X(\omega)}[t-\varepsilon,t) = 0$.

Take now in Eq. (3.3) $\varepsilon = L^{-1/2}$. It then follows that for each t the random variable $|\int_{[0,t)} g(L(t-s))\,\mathrm{d}X_s|$ tends to zero a.s. Furthermore, the right-hand side of the inequality (3.3) is bounded by $2(W_X[0,T) + X_T^*)V(g;[0,\infty))$, which is integrable over

$[0, T]$ with respect to $\|Z\|$. Then $Y_T^L(X, Z) \to 0$ a.s. for $L \to \infty$ by an application of Lebesgue's dominated convergence theorem.

We proceed with the second term $Y_T^L(X, M)$. First we notice that for fixed L the process $Y^L(X, M)$ is a locally square integrable martingale. This can be seen as follows. It is easy to see that the right-hand side of inequality (3.3) is bounded by $6X_t^* V(g; [0, \infty))$. Furthermore, the assumption that X has bounded jumps entails that X^* is a locally bounded process, and a fortiori the same holds for $\int_{[0,\cdot)} g(L(\cdot - s))\, dX_s$, so (being predictable) it belongs to $L^2_{\text{loc}}(M)$; (see Jacod and Shiryaev, 1987, p. 48). Hence, the predictable variation of $Y^L(X, M)$ at T is given by

$$\langle Y^L(X, M)\rangle_T = \int_{[0,T]} \left| \int_{[0,t)} g(L(t-s))\, dX_s \right|^2 d\langle M\rangle_t .$$

As above one can show that $\langle Y^L(X, M)\rangle_T \to 0$ for $L \to \infty$ a.s. by applying the dominated convergence theorem. Hence, a simple application of Lenglart's inequality (cf. Jacod and Shiryaev, 1987, p. 35) yields that $Y_T^L(X, M)$ tends to zero in probability as $L \to \infty$. This completes the proof. □

Some examples of distributions for which the conditions on g in Theorem 3.2 are satisfied are the triangular distribution, the double exponential distribution, the normal distribution (see Table 1 of Feller (1971, p. 503), or the distribution which has the Epanechnikov kernel as its density. (This kernel enjoys some optimality properties in problems of kernel density estimation; see e.g. Van Es (1991, p. 21.) The characteristic function of the uniform distribution on $[-1, +1]$ is not of bounded variation over $[0, \infty)$.

Remark. It is instructive to see that for deterministic times T in the situation where moreover X is a square integrable martingale with deterministic predictable variation, the proof of the above theorem is much simpler and that we do not need that g is of bounded variation (as well as in Proposition 2.1). Indeed consider again Y^L with its quadratic variation given by

$$\langle Y^L\rangle_T = \int_{[0,T]} \left(\int_{[0,t)} g(L(t-s))\, dX_s \right)^2 d\langle X\rangle_t .$$

Taking expectations yield

$$E\langle Y^L\rangle_T = \int_{[0,T]} \int_{[0,t)} g(L(t-s))^2\, d\langle X\rangle_s\, d\langle X\rangle_t .$$

Using again the dominated convergence theorem, we see that $E\langle Y^L\rangle_T$ tends to zero for $L \to \infty$. So $Y_T^L \to 0$ in probability, in view of Chebychev's inequality.

4. Arbitrary semimartingales

In this section we prove the analog of Proposition 3.2 for an arbitrary semimartingale X.

Theorem 4.1. *Let X be a semimartingale and let the function g be of bounded variation over $[0,\infty)$. Let T be a finite stopping time. Then*

$$E_\xi I_T(X; L\xi) \to [X]_T$$

in probability, for $L \to \infty$.

Proof. Decompose X as $X = \check{X} + \hat{X}$, where $\check{X}_\cdot = \sum_{s \leq \cdot} \Delta X_s 1_{\{|\Delta X_s| > 1\}}$. Observe that $\check{X}$ is a process of bounded variation, and that $\hat{X}$ is a process with bounded jumps. Now we can write Y_T^L (see Eq. (3.1)) as the sum of the following four quantities:

$$Y_T^L(\check{X}, \check{X}) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}\check{X}_s\,\mathrm{d}\check{X}_t, \tag{4.1}$$

$$Y_T^L(\check{X}, \hat{X}) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}\check{X}_s\,\mathrm{d}\hat{X}_t, \tag{4.2}$$

$$Y_T^L(\hat{X}, \check{X}) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}\hat{X}_s\,\mathrm{d}\check{X}_t, \tag{4.3}$$

$$Y_T^L(\hat{X}, \hat{X}) = \int_{[0,T]} \int_{[0,t)} g(L(t-s))\,\mathrm{d}\hat{X}_s\,\mathrm{d}\hat{X}_t. \tag{4.4}$$

It follows from Proposition 2.1 that $Y_T^L(\check{X}, \check{X})$ a.s. converges to zero, and from Proposition 3.2 it follows that $Y_T^L(\hat{X}, \hat{X})$ converges to zero in probability as $L \to \infty$. So we only have to focus our attention on the other two terms. Consider first (4.2). Let the T_i be the (finitely many) jump times of $\check{X}$ that are strictly less than T. Then

$$Y_T^L(\check{X}, \hat{X}) = \int_{[0,T]} \sum_{T_i < t} g(L(t-T_i))\Delta X_{T_i}\,\mathrm{d}\hat{X}_t = \sum_{T_i} \int_{(T_i,T]} g(L(t-T_i))\,\mathrm{d}\hat{X}_t \Delta X_{T_i}. \tag{4.5}$$

Fix i and consider one of the summands in Eq. (4.5). As in the proof of Lemma 3.1 we use partial integration and a splitting of the integration interval into $(T_i, T_i^\varepsilon]$ and $(T_i^\varepsilon, T]$, where $T_i^\varepsilon = (T_i + \varepsilon) \wedge T$. So,

$$\begin{aligned}
\int_{(T_i,T]} g(L(t-T_i))\,\mathrm{d}\hat{X}_t &= \hat{X}_T g(L(T-T_i)) - \hat{X}_{T_i} - \int_{(T_i,T]} \hat{X}_t\,\mathrm{d}g(L(t-T_i)) \\
&= \hat{X}_T g(L(T-T_i)) - \hat{X}_{T_i} g(L(T_i^\varepsilon - T_i)) \\
&\quad - \int_{(T_i,T_i^\varepsilon]} (\hat{X}_t - \hat{X}_{T_i})\,\mathrm{d}g(L(t-T_i)) \\
&\quad - \int_{(T_i^\varepsilon,T]} \hat{X}_t\,\mathrm{d}g(L(t-T_i)).
\end{aligned} \tag{4.6}$$

For the absolute value of the first integral on the right-hand side of Eq. (4.6) we have the bound

$$W_{\hat{X}}[T_i, T_i^\varepsilon]\, V(g;\, [0,\infty))$$

and hence also the larger bound

$$W_{\hat{X}}[T_i, T_i + \varepsilon]\, V(g;[0, \infty)),$$

whereas for the second one the bound $\hat{X}_T^* V(g;[L(T_i^\varepsilon - T_i), \infty))$ is valid. Notice that the integrals involved are defined pathwise, so that the estimates above also makes sense for the random times that are figuring here.

Similar to what we did in the proof of Proposition 3.2 we observe that $W_{\hat{X}}[T_i, T_i + \varepsilon]$ tends to zero a.s. if $\varepsilon \to 0$ by right continuity of $\hat{X}$ at T_i. Furthermore, $T_i^\varepsilon - T_i = \varepsilon 1_{\{T_i \le T - \varepsilon\}} + (T - T_i) 1_{\{T_i > T - \varepsilon\}}$ and notice that the last indicator becomes zero if $\varepsilon \to 0$, since all the T_i are strictly less than T. By taking as before $\varepsilon = L^{-1/2}$ all the summands in Eq. (4.5) tends to zero a.s. for $L \to \infty$, and since their number is finite we obtain the same for $Y_T^L(\check{X}, \hat{X})$.

Finally, we look at Eq. (4.3). From Lemma 3.1 we get that

$$|Y_T^L(\hat{X}, \check{X})| \le 2 \sum_{T_i} \{W_{\hat{X}}[T_i - \varepsilon, T_i) V(g;[0, \infty)) + \hat{X}_T^* V(g;[L\varepsilon, \infty))\} |\Delta X_{T_i}|.$$

Application of the same arguments as before shows that also $Y_T^L(\hat{X}, \check{X})$ tends to zero a.s. for $L \to \infty$. This completes the proof. □

5. Some consequences

As a simple consequence of Theorem 4.1 we obtain a representation result for the optional quadratic covariation of two semimartingales.

Let X and Y be arbitrary real valued semimartingales, T a finite stopping time and g be of bounded variation. Define the *cross periodogram* of X and Y for each real number λ by

$$I_T(X, Y; \lambda) = \int_{[0,T]} e^{i\lambda t}\, dX_t \int_{[0,T]} e^{-i\lambda t}\, dY_t.$$

Let ξ be a real random variable as before. Then we have the following corollary.

Corollary 5.1. *Under the conditions of Theorem* 4.1 *we have*

$$E_\xi I_T(X, Y; L\xi) \to [X, Y]_T$$

in probability.

Proof. It is easy to verify that the following form of the polarization formula holds:

$$I_T(X, Y; \lambda) + I_T(Y, X; \lambda) = \tfrac{1}{2}[I_T(X + Y; \lambda) - I_T(X - Y; \lambda)].$$

Then an application of Theorem 4.1 together with the known polarization formula for the square bracket process and the observation that $E_\xi I_T(X, Y; L\xi)$ is real yields the result. □

Remark. One can define the periodogram for a multivariate semimartingale X with values in $\mathbb{R}^n$ as

$$I_T(X; \lambda) = \int_{[0,T]} e^{i\lambda t}\, dX_t \left(\int_{[0,T]} e^{i\lambda t}\, dX_t \right)^*.$$

Then the parallel statement of Theorem 4.1 holds in view of Corollary 5.1 with $[X]$ then $n \times n$-matrix valued optional quadratic variation process.

We end this section with a consequence of Theorem 4.1 in terms of Dirac delta approximations. Return to real valued semimartingales X and let ξ be as in the introduction and assume that $E\xi^2 < \infty$. Then g is twice continuously differentiable, so we obtain from Eq. (1.2),

$$E_\xi I_T(X; L\xi) = X_T^2 - 2X_T \int_{[0,T]} X_t \frac{\partial}{\partial t} g(L(T-t))\,\mathrm{d}t + \int_{[0,T]}\int_{[0,T]} X_t X_s \frac{\partial^2}{\partial t \partial s} g(L(t-s))\,\mathrm{d}t\,\mathrm{d}s. \tag{5.1}$$

The idea is that both the two kernels in Eq. (5.1) behave as a Dirac distribution (although not quite). More precisely, we have the following proposition.

Proposition 5.2. *Let X be a real semimartingale, T a finite stopping time and g a twice continuously differentiable real characteristic function, which is assumed to be of bounded variation over $[0, \infty)$.*
The following statements hold almost surely, respectively, in probability

(i) $\int_{[0,T]} X_t (\partial/\partial t) g(L(T-t))\,\mathrm{d}t \to X_{T-}$

(ii) $\int_{[0,T]}\int_{[0,T]} X_t X_s (\partial^2/\partial t \partial s) g(L(t-s))\,\mathrm{d}t\,\mathrm{d}s \to X_{T-}^2 + [X]_{T-}$

Proof. (i) follows by a similar argument as used in the proof of inequality (3.3) and the choice $\varepsilon = L^{-1/2}$. Indeed, consider the difference (the integral does not change if we replace the integration interval with $[0, T)$)

$$\int_{[0,T)} X_t \frac{\partial}{\partial t} g(L(T-t))\,\mathrm{d}t - X_{T-} = \int_{[0,T-\varepsilon)} X_t \,\mathrm{d}g(L(T-t)) + \int_{[T-\varepsilon,T)} (X_t - X_{T-})\,\mathrm{d}g(L(T-t)) - X_{T-}g(L\varepsilon). \tag{5.2}$$

The absolute value of the first integral in (5.2) is bounded by $X_T^* V(g; [L\varepsilon, \infty))$, whereas the absolute value of the second integral in (5.2) is majorized by $W_X[T-\varepsilon, T)V(g; [0, \infty))$. By choosing $\varepsilon = L^{-1/2}$, we see that all three quantities in (5.2) tend to zero a.s. for $L \to \infty$.

(ii) is then a consequence of (i) and Theorem 4.1. □

Remark. The second statement of this proposition is at first glance perhaps somewhat surprising, since one would expect for continuous X the term X_T^2 only. The extra term $[X]_T$ is due to the fact that X is in general not of bounded variation.

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